Win-or-Nothing Decision Analysis

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Abstract

Decision analysis (DA) is a discipline capable of robustly analyzing many types of decisions. In this paper, win-or-nothing problems will be formulated through a coin tossing problem and a general approach on how a decision-maker should value a win-or-nothing lottery will be presented. Then, binary option prices will show the behavior of a range of win-or-nothing contracts with different probabilities of success.

Keywords: Decision Analysis, Binary option, expected utility, nadex

1. Introduction

A win-or-nothing lottery has an outcome of either success or nothing. For a successful outcome, the gambler wins a predetermined payout, *w*. Otherwise, the gambler walks away with nothing. The cost of the gamble is *c*, any real number. The gamble can also be considered from the short end, where the outcome is either failure or nothing. In this situation, the gambler requires the cost, *c*, to accept the risk of losing *w*. Every contract sold has two sides, win-or-nothing and lose-or-nothing. A decision-maker can buy or sell a contract. Below are some examples of these contracts. Situations 1 and 2 will be analyzed later in this paper, and the others can be analyzed similarly.

1. Flip of a two sided coin. The coin can be biased.
2. Binary option which results in a payoff of either one or zero dollars.
3. Lottery ticket.
4. Medical procedure that has probability p of success and no side effects.
5. Project or research (e.g. oil company researching and drilling for oil).
6. Insurance on whether an event will happen.
7. Bid that can be accepted or rejected. The bid can have a cost.

A normative decision-maker accepts the lottery if it fits their risk profile. The decision-maker must first formulate the decision problem. All the constants must be known and all the possible outcomes must be considered. The mathematical formulation of a win-or-nothing decision and a lose-or-nothing decision are shown below.

$$Payoff_{\text{win-or-nothing}} = \begin{cases} W - C & success \\ -C & otherwise \end{cases}$$

$$Payoff_{\text{lose-or-nothing}} = \begin{cases} -W + C & failure \\ C & otherwise \end{cases}$$

And it follows that the two contract types are equal and opposite; they sum to zero. A portfolio made of both the win-or-nothing and the lose-or-nothing contract is worth zero.

$$Payoff_{\text{lose-or-nothing}} = -Payoff_{\text{win-or-nothing}}$$

$$Payoff_{\text{lose-or-nothing}} + Payoff_{\text{win-or-nothing}} = 0$$

The outcome of a lottery is random. Its probability of success, *p*, is not necessarily known. In fact, different people can infer different probabilities for the same lottery. Howard (1970) described how to infer probability based on limited information using Bayesian updating. Chow and Liu (1968) created multi-attribute probability distributions with limited information. Probabilities can also change with time. Stochastic calculus can be used to find analytical solutions while monte carlo simulation can give numerical solutions of complex events. In this paper, probabilities are either assumed or known from market data. Finally, because probability can change with time, the contract is only considered for a specific moment in time.

For analysis purposes, only monetary costs and payouts will be considered. Variables *c* and *w* will be in dollars. Also, only individual contracts will be considered. Refer to Engels (2005) for analysis of portfolios of independent and dependent binary options.

2. A Coin Tossing Problem

Win-or-nothing decisions can be framed using a coin tossing problem. Suppose a coin has probability p of landing heads. The cost of the gamble is c and the payout from success is w. The decision-maker has a choice of whether or not to take the gamble. We will find that the agent will only take the bet at or below a certain finite cost. To assist in formulating the decision, a decision tree is shown in figure 1.

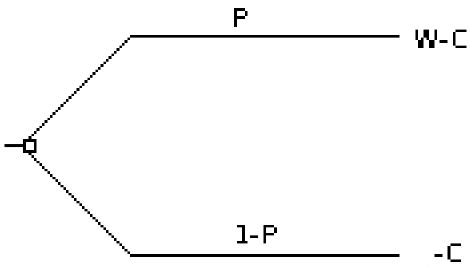

Figure 1. Decision tree for the coin flip

In order to describe the general approach on how to value this contract, we must introduce the concept of utility, or satisfaction attained from a commodity. The marginal satisfaction can change as more commodities are procured. Also, utility can differ among people with different wealth and risk preferences. A utility profile is determined based on personal preferences. This profile is mapped by a utility curve (u-curve). Different requirements can be placed on a u-curve, for example, it is assumed that u-curves are monotonically increasing with money for people who prefer more money to less money. To make consistent decisions, a decision-maker should follow his personal u-curve, or an agent should follow a company's u-curve. To make normative decisions, a decision-maker should maximize expected utility (EU) as per von Neumann-Morgenstern (1944) utility theory. Using the decision tree in figure 1, the decision-maker's expected utility from a lottery is:

$$EU = p\,U(w-c) + (1-p)\,U(-c)$$

The certainty equivalent, $\tilde{x}$, is the maximum dollar amount the decision-maker is willing to trade for the risk. Equivalently, the decision-maker is indifferent between taking a specific risk and receiving the corresponding certainty equivalent. The certainty equivalent is found by taking the inverse of the utility function and substituting in the expected utility. This method is shown in table 1. It can also be found intuitively from the u-curve as shown in figure 2; the decision-maker can reflect the expected utility off the u-curve to find the certainty equivalent.

Table 1: Inverse of EU to find certainty equivalent

	Discrete	*Continuous*
$EU =$	$\sum P_i U_i$	$\int U(x)\,dF(x)$
$EU =$	$U(\tilde{x})$	
$\tilde{x} =$	$U^{-1}(EU)$	

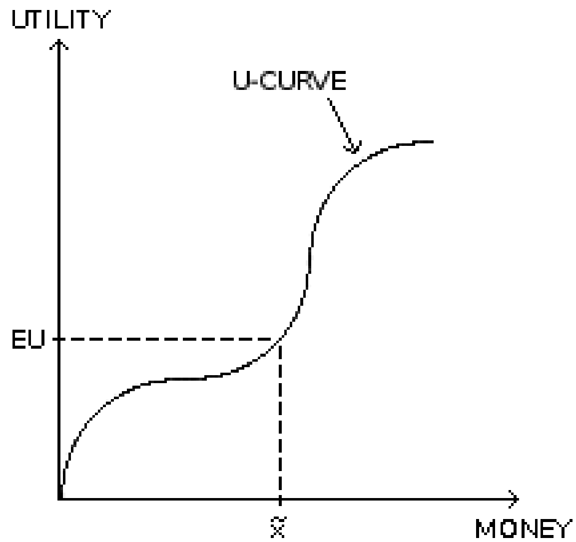

Figure 2: An arbitrary U-curve with corresponding expected utility and certainty equivalent.

A normative and consistent decision-maker will take a gamble only if it corresponds with their utility profile, so

$$\text{if } c \le \tilde{x}, \text{ take the gamble.}$$

Different decision-makers can have different u-curves. For example, given P=0.45, W=20, and C=8, different u-curves are used to find different certainty equivalents. The decision-makers with the second and third u-curves should take the gamble.

Table 2: Sample certainty equivalent calculations

$U(x)$	EU	$\tilde{x}$
x	1	1
x^2	100	10
e^x	73,239	11.20
$1-e^{-2x}$	-4.89×10^6	7.70

For the coin tossing problem, the seller should set their price to be equal to or greater than their certainty equivalent. When sold, they will hold a lose-or-nothing contract. If their risk profile is risk seeking, they will accept low prices. If they are risk averse, they will demand higher prices for taking the same risk. Transactions will occur between a risk-seeking seller and a risk-averse buyer, and vice versa. If a company decides to take more risks, they will be willing to pay more money for the same amount of risk, or take on risk for less money.

3. Binary Options

A binary option closely resembles the coin tossing problem. The difference is that binary options are bought and sold until the expiration date. An exchange can be used as a source of information, assuming market efficiency (see Appendix A). The North American Derivatives Exchange (NADEX) has binary options for Gold. A contract which costs $\$C$ reads: 'Gold >$\$X$ at T' where X is the strike and T is the expiration. If the strike is met at expiration, the contract is worth one hundred dollars, otherwise it is worth zero. Figure 3 shows the payoff of a binary option.

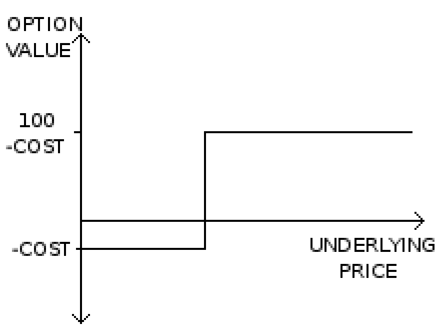

Figure 3. Payoff of a binary call option.

NADEX provides contracts at various strike prices. Table 3 shows the contracts on Gold that expire at 1:30PM at different strikes.

Table 3: NADEX snapshot of gold at 10:53CST on 08/07/09 when gold was valued $955/ounce

Contract	Expiry	Bid Size	Bid	Offer	Offer Size
Gold (Dec) >952.50 at 1:30PM	07-AUG-09	100	94.0	99.0	100
Gold (Dec) >955.00 at 1:30PM	07-AUG-09	100	86.0	91.0	100
Gold (Dec) >957.50 at 1:30PM	07-AUG-09	100	70.0	76.0	100
Gold (Dec) >960.00 at 1:30PM	07-AUG-09	100	47.0	55.0	100
Gold (Dec) >962.50 at 1:30PM	07-AUG-09	100	26.0	33.0	100
Gold (Dec) >965.00 at 1:30PM	07-AUG-09	100	10.0	15.0	1
Gold (Dec) >967.50 at 1:30PM	07-AUG-09	100	2.0	7.0	100

These values can be mapped into a discrete cumulative density function. We normalize and convert the quasi-CDF into a pmf. Then we can find the probability for each range. Similar work has been done by Engels (2005). Her work takes extra unnecessary steps in finding the probability of each range of options. My work was done independently and prior to learning of Engels' work.

$$P(x>X)=F(x)$$

$$F(x)=\sum_{-\infty}^{x} p(x)$$

$$p(x)=\frac{\Delta F}{\Delta x}$$

$$P(960<x\leq 962.5)=0.47-0.26=0.21$$

Similarly for all the ranges, we construct the pmf.

$$P(x\leq 952.5)=.06$$
$$P(952.50<x\leq 955.00)=0.08$$
$$P(955.00<x\leq 957.50)=0.16$$
$$P(957.50<x\leq 960.00)=0.23$$
$$P(960.00<x\leq 962.50)=0.21$$
$$P(962.50<x\leq 965.00)=0.16$$
$$P(965.00<x\leq 967.50)=0.08$$
$$P(967.50<x)=0.02$$

Using the pmf obtained, we infer the probability of gold having a value between 960 and 962.5 at expiration is 0.21. To confirm that this is a probability mass function, we sum the probabilities to confirm it equals unity (.06+ .08+ .16+ .23+ .21+ .16+ .08+ . 02=1). Then we can find whatever characteristics of the distribution we wish, including mean, standard deviation, skew, kurtosis, entropy, etc.

$$\sum_{-\infty}^{\infty} p(x)=1$$

$$\bar{x}=\sum xp(x)=958.38$$

$$\sigma=\sqrt{\left(\frac{\sum (x_i-\bar{x})^2}{N-1}\right)}=4.19$$

$$g_1=\frac{\frac{1}{n}\sum (x_i-\bar{x})^3}{\left(\frac{1}{n}\sum (x_i-\bar{x})^2\right)^{3/2}}=-0.1$$

$$g_2=\frac{\frac{1}{n}\sum (x_i-\bar{x})^4}{\left(\frac{1}{n}\sum (x_i-\bar{x})^2\right)^{2}}-3=-0.49$$

$$H=-\sum p_i\log_2(p_i)=2.67$$

The mean of the distribution is 958.38 and the standard deviation is 4.19. From the skew and kurtosis, this distribution is skewed to the right and has lower peak and smaller tails than the Gaussian distribution. We also find that entropy is 2.67, which can be used to compare distributions Abbas (2006). For discussion of the continuous case, see Appendix B.

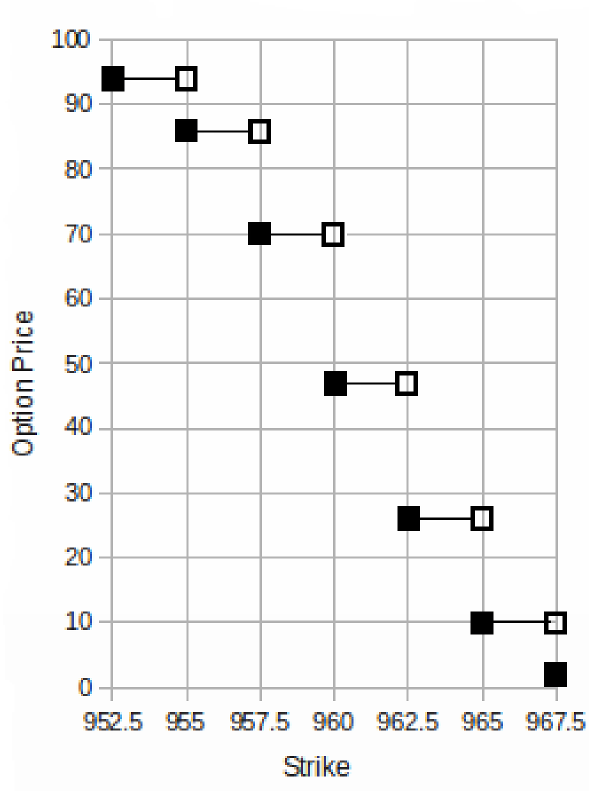

Figure 4. The quasi-cdf of Gold binary options

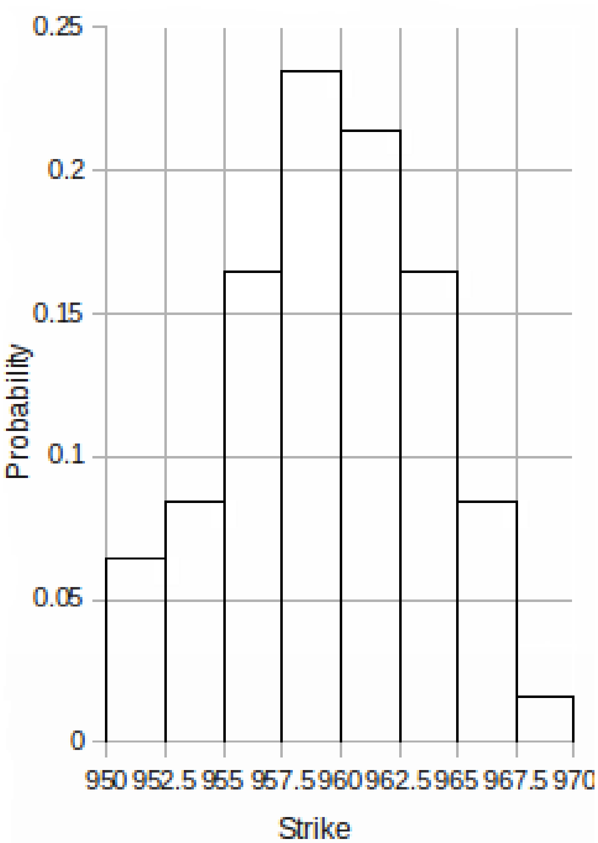

Figure 5. The pmf of Gold binary options

The corresponding cdf and pmf are shown in figures 4 and 5. The pmf can be used as a variable in stochastic models. It can also be used to speculate future price movements. It would be interesting to see this distribution change over time. Now that the win-or-nothing decision is formulated, my next paper in progress investigates risk aversion at various levels of wealth when dealing with win-or-nothing decisions.

4. APPENDIX

APPENDIX A - The legitimacy of binary options markets.

Criticism 1: Low volume, low depth, and wide spreads.
Rebuttle: That can be a problem with any exchanges. And exchanges with high volume, high depth, and small spreads are usually expensive (e.g. CME seats and transaction costs). NADEX answers those concerns by internally maintaining $5 spreads and $10000 in depth for each gold contract.

Criticism 2: Exchanges are manipulated and are not good sources of information.
Rebuttle: This can be a problem with any exchange. But if an arbitrage opportunity exists, someone will likely take it. Serwer's (2005) article in Fortune magazine discusses how Intrade binary options for Saddam Hussein being "captured and neutralized" rose before any news was released. In a free market, people with information will take an arbitrage opportunity. Also, prediction markets can be used within a company to see what employees think of different divisions/projects. Using Prediction Markets to Track Information Flows: Evidence from Google (Cowgill et al, 2008) talks about how Google uses these exchanges to compare the executive and employee risk aversion. It is important for everyone in a company to make decisions with the same risk aversion.

Ideally, a popular exchange will work on the same principals as GNU/GPL. It will be free with open access to everyone. Until then, markets like NADEX are close alternatives.

APPENDIX B – The Continuous Case

To fit the discrete data presented in section three to continuous density function f(x), plug the formula into the following equations. Some equations may be difficult to integrate so I suggest using common distributions with analytical results. Otherwise, use numerical methods. Here are the corresponding formulas in order:

$$P(x > X) = F(x)$$

$$F(x) = \int_{-\infty}^{\infty} f(x)\,dx$$

$$f(x) = \frac{d}{dx} F(x)$$

$$\int_{-\infty}^{\infty} f(x) = 1$$

$$\bar{x} = \int_{-\infty}^{\infty} x f(x)$$

$$\sigma = \int (s - \mu)^2 f(x)\,dx$$

5. References

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